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Hyper-Hamiltonian dynamics

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Abstract

We introduce an extension of Hamiltonian dynamics, defined on hyper-Kahler manifolds, which we call 'hyper-Hamiltonian dynamics'. We show that this has many of the attractive features of standard Hamiltonian dynamics. We also discuss the prototypical integrable hyper-Hamiltonian systems, i.e. quaternionic oscillators.

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Introduction

The description provided by Hamiltonian dynamics applies to many fields of physics; due to its rich geometrical structure, it is also very convenient and indeed widely used whenever possible.

Hamiltonian formalism is based on symplectic structures; a special but relevant class of symplectic manifolds is provided by Kahler manifolds. In fact, in any symplectic manifold M we can give locally (and globally if M is contractible, e.g. $M = \boldsymbol{R}^{2n}$) a complex structure associated with the symplectic one and, by the introduction of a suitable metric, a Kahler structure.

In relatively recent years, mathematicians on the one hand, and (theoretical and mathematical) physicists on the other, have become interested in a special kind of Kahler structure, i.e. *hyper-Kahler* structures [8]. These are Kahler with respect to three different complex structures, having such relations among them that, roughly speaking, they can be seen as the complex structures associated with the three independent imaginary units of the quaternion field.

A Riemannian manifold (necessarily of dimension $4n$) equipped with a hyper-Kahler structure is called a *hyper-Kahler manifold*. These turn out to be very interesting from the point of view of geometry [20, 31, 33], and also relevant in the description of (non-Abelian) monopoles [7, 9, 26]; they also bear a close connection, which we shall not discuss here, with twistors [8, 30, 35] and thus in particular to interesting classes of integrable systems [17, 28]. It has also been realized that hyper-Kahler (and quaternionic-Kahler [2]) manifolds are relevant in supersymmetry and supergravity theories and related to sigma-models (see e.g. [16, 19, 27]). Canonical examples of hyper-Kahler manifolds are quaternion linear spaces $\boldsymbol{H}^n \approx \boldsymbol{R}^{4n}$ and cotangent bundles of (special) Kahler manifolds [19, 34]; a specially fruitful method of constructing nontrivial hyper-Kahler manifolds is through a generalization of the Marsden–Weinstein momentum map [27]. For an overview of recent results in quaternionic and hyper-Kahler geometry (not needed in the present work), the reader is referred to [20].

It is obvious that a hyper-Kahler structure can be described in symplectic terms; we speak then of a *hypersymplectic structure*. It is remarkable, and came much to our surprise, that it is possible to provide a generalization of Hamilton mechanics based on such a hypersymplectic structure, which we do here. What is more relevant is that this *hyper-Hamiltonian dynamics* retains most of the appealing features of standard Hamiltonian mechanics, as we show in this paper.

Our initial motivation was provided by integrable systems. We shall consider a special class of these, i.e. *quaternionic oscillators* (see section 7), which we expect to be the paradigm of a nontrivial integrable hyper-Hamiltonian system.

Limiting ourselves to considering systems with compact energy manifolds, a $2m$-dimensional Hamiltonian integrable system can be described by means of suitable real (action-angle) coordinates (I_a, φ_a), or more compactly complex coordinates $z_a = I_a \exp[\mathrm{i}\varphi_a]$, so that its evolution is described by m constant complex rotations, $\dot{z}_a(t) = \mathrm{i}\omega_a z(t)$.

In slightly different (but equivalent) terms, we can use coordinates (I_a, g_a) where $g_a \in G = U(1)$; here of course $g_a = \exp[\mathrm{i}\varphi_a]$, so we are just describing action angle coordinates in group-theoretical terms, using the isomorphism $S^1 \simeq \boldsymbol{C}_1 \simeq U(1)$ (here $\boldsymbol{C}_1$ are the complex numbers of unit norm). In this language, the evolution is described by $\mathrm{d}I_a/\mathrm{d}t = 0$, $\mathrm{d}g_a/\mathrm{d}t = \gamma_a$, constant elements of the Lie algebra $u(1)$; indeed, as well known, the isomorphism $S^1 \simeq \boldsymbol{C}_1 \simeq U(1)$ identifies the Lie algebra of $U(1)$ with imaginary numbers.

As discussed below, a $4n$-dimensional hyper-Hamiltonian integrable system can be described by real (spin) coordinates $(I_a; g_a)$, where $g_a \in G = SU(2)$. Its evolution is accordingly described by $\mathrm{d}I_a/\mathrm{d}t = 0$, $\mathrm{d}g_a/\mathrm{d}t = \gamma_a$, constant elements of the Lie algebra $su(2)$.

As well known, an integrable Hamiltonian system in a $2m$-dimensional phase space M is associated with a fibration of M in tori $\boldsymbol{T}^m$. In the case of integrable hyper-Hamiltonian systems in a $4k$-dimensional phase space, we shall have a fibration not in tori $\boldsymbol{T}^{2k} = S^1 \times \cdots \times S^1 \equiv U(1) \times \cdots \times U(1)$, but rather in manifolds $\boldsymbol{V}^k = S^3 \times \cdots \times S^3 \equiv SU(2) \times \cdots \times SU(2)$.

1. Quaternionic symplectic structures

In this section we recall the basic geometric definitions to be used in the following. These will mainly concern hyper-Kahler and quaternionic geometry; for a short introductions to these the reader is referred to [8, 13], while more details on these are provided for example by [2, 7, 9, 12, 17, 26, 31, 33]. Symplectic geometry is discussed for example in [5, 14, 18, 24]; for Hamiltonian mechanics see for example [1, 3, 6].

We preliminarily recall that if M is a $2m$-dimensional manifold equipped with a (Riemannian) metric g, a *complex structure* on M is a $(1, 1)$ type tensor field Y such that $Y^2 = -I$, which is covariant constant; a *symplectic form* ω on M is a non-degenerate and closed two-form. Then $\omega(v, w)$ can be written as $g(v, Jw)$ for some $(1, 1)$-type antisymmetric tensor field on M. If this J is orthogonal for g, we say that ω is compatible with the metric (or briefly g-compatible), or equivalently that ω is *unimodular*. If this is the case, then $(1/m!)\omega \wedge \cdots \wedge \omega = s\Omega$, with $s = \pm 1$ and Ω the volume form on M. We say accordingly that unimodular symplectic forms are of positive or negative type.

We also recall that M is a *Kahler manifold* if it is equipped with a metric g, a complex structure Y and a symplectic form ω, satisfying the *Kahler relation* $\omega(v, w) := g(v, Yw)$, or equivalently $g(v, w) = \omega(Yv, w)$.

In local coordinates, if $Y^i{}_j(x)$ describes the complex structure, the associated symplectic form is given by $\omega = (1/2)K_{ij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j$ with $K_{ij}(x) = g_{im}(x)Y^m_j(x)$.

We can pass to consider hyper-Kahler manifolds. Now—and always in the following—M will be a smooth $4n$-dimensional real manifold endowed with a Riemannian metric g, and ϵ will denote the completely antisymmetric (Levi-Civita) symbol.

Definition 1. *A* hypercomplex structure *on M is an ordered triple* $\boldsymbol{Y} = (Y_1, Y_2, Y_3)$ *of complex structures on M satisfying* $Y_\alpha Y_\beta = \epsilon_{\alpha\beta\gamma} Y_\gamma - \delta_{\alpha\beta} I$. *If the* Y_α *are orthogonal complex structures on* (M, g)*, we say that* $\boldsymbol{Y}$ *is orthogonal.*

Remark 1. The Y_α making up a hypercomplex structure satisfy the quaternionic relations. In Lie algebraic terms, the $\tilde{Y}_\alpha = (1/2)Y_\alpha$, which satisfy $[\tilde{Y}_\alpha, \tilde{Y}_\beta] = \epsilon_{\alpha\beta\gamma}\tilde{Y}_\gamma$, realize the $su(2)$ algebra.

Definition 2. *A* hyper-Kahler structure *on M is a quadruple* (g, Y_1, Y_2, Y_3)*, where g is a metric on M, the* Y_α *are an orthogonal hypercomplex structure* (M, g) *and the two-forms* ω_α *defined by the complex structures* Y_α *via the Kahler relation are closed and nondegenerate on* M.

Notice that the forms ω_α are therefore (independent) symplectic forms on M. As dealing with differential forms is equivalent to—but rather more convenient in practice—than dealing with (1, 1) tensor fields, we shall generally find it more convenient to focus on these.

Definition 3. *A* hypersymplectic structure $\boldsymbol{O} = \{\omega_1, \omega_2, \omega_3\}$ *on the Riemannian manifold* (M, g) *is an ordered triple of g-compatible symplectic structures on M, such that the complex structures* Y_α *defined by the* ω_α *via the Kahler relation are a hypercomplex structure on* M.

If $\boldsymbol{O}$ is a hypersymplectic structure on (M, g), the linear span (with real coefficients) of the ω_α is the real linear space $\mathcal{Q} := \{\sum_\alpha c_\alpha \omega_\alpha\} \subset \Lambda^2(M)$. This is called the *quaternionic symplectic structure* generated by the hypersymplectic structure $\boldsymbol{O}$, and $\boldsymbol{O}$ is an *admissible basis* for $\mathcal{Q}$ [2]. The unit sphere in $\mathcal{Q}$ (with the natural metric, see below) will be denoted as $\mathcal{S}$ ($\mathcal{S} \approx S^2$).

Remark 2. The $\mathcal{S}$ defined above is related to the *twistor space* on M [7–9, 26, 30, 35].

The natural scalar product in $\mathcal{Q}$ (seen as a linear space) between $q_1 = a_\alpha\omega_\alpha$ and $q_2 = b_\alpha\omega_\alpha$ is $(q_1, q_2) := a_\alpha b_\alpha$. If we choose a local coordinate system, and we associate with q_i the matrices Q_i, this coincides with the natural scalar product in the linear space $\boldsymbol{Q}$ generated by the complex structures $\{Y_\alpha\}$, i.e. $(Q_1, Q_2) := (4n)^{-1}\,\mathrm{Tr}(Q_1^\dagger Q_2)$.

Lemma 1. *Any nonzero* $\omega \in \mathcal{Q}$ *is a symplectic structure on M. If* $\omega \in \mathcal{Q}$*, then* ω *is unimodular (and thus defines a Kahler structure in M) if and only if* $\omega \in \mathcal{S}$.

Proof. The first part is trivial. As for the second, if $\omega = c_\alpha\omega_\alpha \in \mathcal{Q}$, then ω yields the complex structure $Y = c_\alpha Y_\alpha$, where the $\{Y_\alpha\}$ are the hyper-Kahler structure (so that, in particular, $\{Y_\alpha, Y_\beta\} = -2\delta_{\alpha\beta}$). We have therefore $Y^2 = -Y^T Y = -(\sum_\alpha c_\alpha^2)I$, and hence the statement. □

The three symplectic structures ω_α can be seen as associated with the imaginary units of the quaternions; it is thus natural that if we operate a (pure imaginary) rotation in the quaternions, we obtain three different symplectic structures which still generate the same quaternionic structure. In other words, we can change the basis in $\mathcal{Q}$ preserving the quaternionic relations, i.e. passing to a different admissible basis. Notice that in this case the sphere $\mathcal{S} \subset \mathcal{Q}$ is invariant.

Definition 4. *Two hypersymplectic structures* $(\boldsymbol{O}, g)$ *and* $(\hat{\boldsymbol{O}}, g)$ *on* M*, spanning the same quaternionic symplectic structure are said to be* equivalent.

Remark 3. More generally, consider a map $\Phi : M \to M$ and let Φ^* be its pullback; if we consider local coordinate $\{x^i\}$ on M, we can write $\omega = (1/2)K_{ij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j$; then $\Phi^*(\omega) = (1/2)(A^+KA)_{ij}(x)\,\mathrm{d}x^i \wedge \mathrm{d}x^j$, where $A = (D\Phi)$ is the Jacobian of Φ. Thus if $A(x) \in O(4n, \boldsymbol{R})$, then Φ^* is a morphism of hyper-Kahler structures of (M, g); if $A^+\boldsymbol{Q}A = \boldsymbol{Q}$, then Φ^* maps $\mathcal{Q}$ into itself (and necessarily preserves $\mathcal{S}$), i.e. maps $\boldsymbol{O}$ to an equivalent hypersymplectic structure.

2. Equations of motion

In this section we define a class of equations of motion in a hyper-Kahler manifold; these are associated with the hyper-Kahler structure and define a Liouville dynamics on the manifold.

Let us consider a hyper-Kahler manifold (M, g, Y_1, Y_2, Y_3) of real dimension $4n$. This can be equivalently seen as a hypersymplectic manifold (M, g, ω_α) with ω_α the symplectic forms associated with Y_α via g; in the following we shall refer to the hyper-Kahler structure even when we focus on the symplectic aspect. The symbol s will have value ± 1, depending whether we are considering positive or negative type symplectic forms ω_α. We define, for ease of notation, $\zeta_\alpha = \omega_\alpha \wedge \cdots \wedge \omega_\alpha$ (with $2n-1$ factors).

Any triple of smooth functions $\mathcal{H}^\alpha : M \to \boldsymbol{R}$ $(\alpha = 1, 2, 3)$ defines a vector field $X : M \to \mathrm{T}M$ by the equations of motion

$$X \lrcorner \Omega = \frac{1}{(2n-1)!} \sum_{\alpha=1}^{3} \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha. \tag{1}$$

We call this the *hyper-Hamiltonian vector field* on $(M, g, \boldsymbol{O})$ associated with the triple $\mathcal{H}^\alpha$.

We stress that the vector field X is uniquely defined by this. Also note that, for any α, one obtains $\omega_\alpha \wedge \cdots \wedge \omega_\alpha = [(2n)!]\, s\, \Omega$ (the ω_α involved in the wedge product are $2n$). Using this relation, the equations of motion can also be rewritten as

$$X \lrcorner \sum_{\alpha=1}^{3} \omega_\alpha \wedge \zeta_\alpha = (6sn) \sum_{\alpha=1}^{3} \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha.$$

Lemma 2. *Equation (1) defines a Liouville vector field on* M*, i.e.* $\mathcal{L}_X(\Omega) = 0$.

Proof. By the general definition of Lie derivative, $\mathcal{L}_X(\Omega) = \mathrm{d}(X \lrcorner \Omega) + X \lrcorner (\mathrm{d}\Omega)$, and obviously $\mathrm{d}\Omega = 0$. If X satisfies (1) we have therefore $\mathcal{L}_X(\Omega) = \sum_\alpha \mathrm{d}(\mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha)$, which is zero since the forms ζ_α are closed. □

Lemma 3. *The vector field X defined by (1) can be rewritten as* $X = X_1 + X_2 + X_3$*, where* X_α *satisfies* $X_\alpha \lrcorner \omega_\alpha = \mathrm{d}\mathcal{H}^\alpha$ *for* $\alpha = 1, 2, 3$.

Proof. It is immediate to check that $X = \sum_\alpha X_\alpha$ with X_α defined as above yields a solution to (1). As X is uniquely defined by (1), this proves the statement. □

Remark 4. It follows from these that if $\omega_\alpha = (1/2)K^{(\alpha)}_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j$, then the hyper-Kahler vector field (1) can be written as $X = f^i \partial_i$, where $f^i = \sum_\alpha K^{ij}_\alpha \nabla_j \mathcal{H}^\alpha$.

Remark 5. If we have a hyper-Kahler manifold (M, g) and a Hamiltonian vector field X (with respect to a symplectic structure ω being part of a hypersymplectic structure $\boldsymbol{O}$), this can obviously be seen as a hyper-Hamiltonian vector field just by choosing two of the $\mathcal{H}^\alpha$ to be constant. One could wonder whether all the hyper-Hamiltonian vector fields on M can be Hamiltonian by a suitable choice of a symplectic structure; this is not the case even in the simplest setting ($M = \boldsymbol{R}^4$), as we show by explicit example in lemma 5 (see section 6).

The hyper-Hamiltonian vector field in M induces a vector field (which we also call hyper-Hamiltonian) in extended phase space, i.e. in $M \times \boldsymbol{R}$, where the $\boldsymbol{R}$ space has coordinate t (and represents the time).

If we introduce local coordinates $\{x^1, \ldots, x^{4n}, t\}$ in $M \times \boldsymbol{R}$, the dynamics in $M \times \boldsymbol{R}$ will be described by a vector field $Z = z^0(x, t)\partial_t + z^i(x, t)\partial_i$ (here and in the following we write ∂_i for $\partial/\partial x^i$, $\partial_t = \partial/\partial t$). The equations of motion given above are equivalent to defining the vector field Z to be $Z = \partial_t + X$, where obviously X is defined by (1).

Due to the closeness of ω_α, we can locally find one-forms σ_α such that $\omega_\alpha = \mathrm{d}\sigma_\alpha$, and locally define forms φ, ϑ in $\Lambda^{(4n-1)}(M \times \boldsymbol{R})$ given by

$$\varphi = \sum_{\alpha=1}^{3} \sigma_\alpha \wedge \zeta_\alpha, \qquad \vartheta = \varphi + (6sn) \sum_{\alpha=1}^{3} \mathcal{H}^\alpha\, \zeta_\alpha \wedge \mathrm{d}t. \tag{2}$$

Note, for later use, that the $(4n)$-form $\mathrm{d}\vartheta$ is nonsingular, and that $\mathrm{d}\varphi$ is proportional to the volume form Ω.

When ω_α is exact (in particular if M has vanishing second cohomology group, for example for $M = \boldsymbol{R}^{4n}$), the σ_α and related forms are globally defined. In order to avoid repeating too frequently that the considerations to be presented are local, we shall assume from now on that the ω_α are exact.

Theorem 1. *Let M be a hyper-Kahler manifold, and let $\mathcal{H}^\alpha : M \to \boldsymbol{R}$ ($\alpha = 1, 2, 3$) be assigned smooth functions; let ϑ be the form defined by (2). Then the equations of motion (1) are equivalent to*

$$Z \lrcorner\, \mathrm{d}\vartheta = 0, \qquad Z \lrcorner\, \mathrm{d}t = 1 \tag{3}$$

where Z is a vector field on $M \times \boldsymbol{R}$.

Proof. The equation $Z \lrcorner\, \mathrm{d}t = 1$ means that we can write Z in the form $Z = \partial_t + Y$; the other equation $Z \lrcorner\, \mathrm{d}\vartheta = 0$ then yields, with simple algebra and separating forms with and without a $\mathrm{d}t$ factor, two equations:

$$Y \lrcorner \sum_{\alpha=1}^{3} \omega_\alpha \wedge \zeta_\alpha = (6sn) \sum_{\alpha=1}^{3} \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha \qquad \text{and} \qquad Y \lrcorner \sum_{\alpha=1}^{3} \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha = 0.$$

The second of these is a trivial consequence of the first one, but the first is just (1). Thus, as (1) uniquely determines X, we have $Y \equiv X$. $\square$

3. Conservation laws and Poisson-like brackets

For the class of systems defined above we have a natural conserved $(4n-1)$-form Θ, canonically associated with the triple $\{\mathcal{H}^1, \mathcal{H}^2, \mathcal{H}^3\}$ and defined as

$$\Theta = \sum_{\alpha=1}^{3} \mathrm{d}\mathcal{H}^\alpha \wedge \zeta_\alpha. \tag{4}$$

Theorem 2. *Let (M, g, ω_α) be a hyper-Kahler manifold, $\{\mathcal{H}^\alpha\}$ be any triple of functions $\mathcal{H}^\alpha : M \to \boldsymbol{R}$, X be the hyper-Hamiltonian flow defined by (1) and Θ be defined by (4). Then $\mathcal{L}_X(\Theta) = 0$.*

Proof. The form Θ is closed, hence $\mathcal{L}_X(\Theta) = \mathrm{d}(X \lrcorner \Theta)$; the explicit expression of Θ and the equations of motion (1) give

$$(X \lrcorner \Theta) = (2n-1)![X \lrcorner (X \lrcorner \Omega)],$$

which is identically zero since we are contracting an alternating form twice with the same vector. □

Notice that $(4n-1)$ forms χ on M are canonically associated with vector fields Y on M via $Y \lrcorner \Omega = \chi$; we write $Y = F(\chi) = Y_\chi$. There is a natural operation $\{.,.\} : \Lambda^{(4n-1)}(M) \times \Lambda^{(4n-1)}(M) \to \Lambda^{(4n-1)}(M)$, defined as follows. Given forms $\chi, \Psi \in \Lambda^{(4n-1)}(M)$, we consider the associated vector fields Y_χ, Y_Ψ; take the commutator $Y_\Gamma := [Y_\chi, Y_\Psi]$. This defines an associated form $\Gamma \in \Lambda^{(4n-1)}(M)$, and we define $\{\chi, \Psi\}$ to be just Γ. In other words,

$$\{\chi, \Psi\} = F^{-1}([F(\chi), F(\Psi)]). \tag{5}$$

We stress that with this notation, $\Theta = F(X)$ with X the hyper-Hamiltonian vector field. Note also that if we have two conserved $(4n-1)$-forms Θ_i, we can generate another (possibly not independent from these, or zero) conserved form $\{\Theta_1, \Theta_2\}$. In this respect, $\{.,.\}$ is reminiscent of the Poisson brackets of standard Hamiltonian mechanics; however, we have to remark that the situation differs substantially from the standard Poisson brackets, because to define our brackets we do not use the hyper-Kahler structure of the manifold, but only the isomorphism between vector fields and $(4n-1)$-forms induced by the volume form on M.

4. Variational formulation

In this section we shall formulate a local variational principle related to the hyper-Hamiltonian equations of motion introduced in section 2.

In order to state in a geometrical framework our principle, we need to consider a local fibre bundle structure on the hyper-Kahler manifold M. This local fibration allows us to describe a particular class of variations ('vertical' with respect to the fibration, as we make precise in a moment) that generalize isochronous variations considered in the variational principle for standard Hamiltonian mechanics.

Let us consider a hyper-Kahler manifold (M, g, ω_α) of real dimension $4n$, and a triple $\{\mathcal{H}^\alpha\}$ of Hamiltonian functions; we shall consider the extended phase space $M \times \boldsymbol{R}$, which we see as a trivial fibre bundle $t : M \times \boldsymbol{R} \to \boldsymbol{R}$.

In order to properly set the local variational problem in a chart M_i of M, we shall need to consider a double fibration

$$M_i \times \boldsymbol{R} \xrightarrow{\pi_i} B_i \xrightarrow{\tau_i} \boldsymbol{R}$$

where the base manifold B_i of the fibre bundle $\pi_i : M_i \times \boldsymbol{R} \to B_i$ is a manifold of dimension $(4n-1)$, fibred itself over $\boldsymbol{R}$ with projection τ_i. We also require, obviously, that $\tau_i \circ \pi_i = t$ on $M_i \times \boldsymbol{R}$.

For ease of notation, we shall from now on just write M for M_i and B for B_i, i.e. use a 'global' notation. We stress that the double fibration considered here is not a general global construction associated with the geometrical structure of a hyper-Kahler manifold; in any case, this double fibration can be considered locally in M_i for a generic hyper-Kahler manifold M, and the choice of the local base manifold B_i is widely arbitrary. Note that in the simple but relevant case $M = \boldsymbol{R}^{4n}$ the double fibration exists globally.

We denote the sets of sections of the bundles introduced above, respectively, by $\Gamma(\pi)$ and $\Gamma(\tau)$, and similarly for $\Gamma(t)$. We denote by $\mathcal{V}(\pi)$ the set of vertical vector fields for the fibration $\pi : M \times \boldsymbol{R} \to B$.

For $V \in \mathcal{V}(\pi)$, we denote by $\psi_s : M \times \boldsymbol{R} \to M \times \boldsymbol{R}$ the flow generated by V. We want to consider variations of sections[4] $\Phi \in \Gamma(\pi)$ under the action of $V \in \mathcal{V}(\pi)$ [25, 32].

Definition 5. *Let $\Phi \in \Gamma(\pi)$. The* variation *of Φ under the vertical vector field V is the section $\tilde{\psi}_s(\Phi) := \psi_s \circ \Phi \in \Gamma(\pi)$.*

Remark 6. The nature of the double fibration $M \times \boldsymbol{R} \xrightarrow{\pi} B \xrightarrow{\tau} \boldsymbol{R}$, where $\tau \circ \pi = t$, ensures that vertical vector fields $V \in \mathcal{V}$ cannot have components along ∂_t; that is, we are actually considering isochronous variations. We also recall that, in order to consider variation of the section Φ, we do not need a vertical vector field V defined on all $M \times \boldsymbol{R}$, but just a vertical vector field defined along Φ.

The main object to be considered is the $(4n-1)$-form ϑ on $M \times \boldsymbol{R}$, defined by (2). We recall that $\vartheta := \sum_{\alpha=1}^{3} [\sigma_\alpha \wedge \zeta_\alpha + (6ns)\mathcal{H}^\alpha \, \zeta_a \wedge \mathrm{d}t]$, where $\mathrm{d}\sigma_\alpha = \omega_\alpha$.

Let us consider a compact $(4n-1)$-dimensional submanifold with boundary $C \subseteq B$. We define a functional $I : \Gamma(\pi) \to \boldsymbol{R}$ given by

$$I(\Phi) = \int_C \Phi^*(\vartheta) \tag{6}$$

where $\Phi^*(\vartheta)$ denotes, as customary, the pullback of ϑ by Φ.

In the following we shall consider only vertical vector fields $V \in \mathcal{V}(\pi)$ such that V vanish on $\pi^{-1}(\partial C)$, where ∂C is the boundary of C. This is just the familiar condition of zero variation on the boundary of the integration region. We denote these as $\mathcal{V}_C(\pi)$.

Definition 6. *A section $\Phi \in \Gamma(\pi)$ is* extremal for *I if and only if*

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\int_C (\tilde{\psi}_s(\Phi))^*(\vartheta)\right]_{s=0} = 0 \tag{7}$$

whenever $V \in \mathcal{V}_C(\pi)$. In this case we write $(\delta I)(\Phi) = 0$.

Theorem 3a. *A section $\Phi \in \Gamma(\pi)$ is extremal for I defined by (6) if and only if $\Phi^*(V \lrcorner \mathrm{d}\vartheta) = 0$ for all $V \in \mathcal{V}_C(\pi)$.*

Proof. This is a standard theorem of variational analysis (see for example chapter XII of [25]). □

Remark 7. Note that $\mathcal{V}(\pi)$ is two dimensional as a module over the algebra of smooth functions $F : M \times \boldsymbol{R} \to \boldsymbol{R}$. With V_1, V_2 a pair of generators for $\mathcal{V}(\pi)$, the condition $\Phi^*(V \lrcorner \mathrm{d}\vartheta) = 0 \, \forall \, V \in \mathcal{V}_C(\pi)$ can be written as $\Phi^*(V_1 \lrcorner \mathrm{d}\vartheta) = 0 = \Phi^*(V_2 \lrcorner \mathrm{d}\vartheta)$. This is independent of C.

[4] Should the reader be misled by our 'global' notation, we note these are actually local sections $\Phi_i \in \Gamma(\pi_i)$, i.e. $\Phi_i : M_i \to M_i \times \boldsymbol{R}$.

Sections Φ which are extremal for I are related to the hyper-Hamiltonian vector field Z in that Z is the characteristic vector field for Φ, as discussed below.

The relation between I and Z is better understood in the language of ideals of differential forms [15, 22] (some basic definitions used here are recalled in the appendix), which we shall call just ideals for short. With this language, and recalling remark 7, theorem 3a above can be restated as follows:

Theorem 3b. *Let V_1, V_2 generate $\mathcal{V}(\pi)$. A section $\Phi \in \Gamma(\pi)$ is extremal for I defined by (6) if and only if Φ is an integral manifold of the ideal $\mathcal{J}$ generated by $V_1 \lrcorner \, \mathrm{d}\vartheta$ and $V_2 \lrcorner \, \mathrm{d}\vartheta$.*

In view of this fact, we shall say that $\mathcal{J}$ is the ideal associated with the variational principle $\delta I = 0$.

We can now discuss the relation between the vector field Z introduced in section 2 and the variational principle based on I. We shall first establish a simple lemma and an immediate corollary thereof.

Lemma 4. *Let α be a nonzero N-form in the $(N+1)$-dimensional manifold M. Let X, V_1, V_2 be three independent and nonzero vector fields on M. Then $V_1 \lrcorner (X \lrcorner \alpha) = 0 = V_2 \lrcorner (X \lrcorner \alpha) = 0$ implies (and is thus equivalent to) $X \lrcorner \alpha = 0$. Moreover, the space of vector fields Y satisfying $Y \lrcorner \alpha = 0$ is a one-dimensional module over $\Lambda^0(M)$.*

Proof. Choose local coordinates $\{x^0, x^1, \ldots, x^N\}$ in M; we can always take $X = \partial_0$, $V_1 = \partial_1$ and $V_2 = \partial_2$. We write $\Omega = \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^N$; then, in full generality, $\alpha = \sum_{k=0}^N c_k(\partial_k \lrcorner \Omega)$. Now $\partial_1 \lrcorner (\partial_0 \lrcorner \alpha) = 0$ implies $c_j = 0$ for $j \neq 0, 1$ and $\partial_2 \lrcorner (\partial_0 \lrcorner \alpha) = 0$ implies $c_j = 0$ for $j \neq 0, 2$. Imposing both equations yields $\alpha = c_0(\partial_0 \lrcorner \Omega) \equiv c_0(X \lrcorner \Omega)$. This satisfies, of course, $X \lrcorner \alpha = 0$; conversely $Y \lrcorner \alpha = 0$ implies $Y = fX$. □

Corollary 1. *Let α, V_1, V_2 be as above. Then the ideal $\mathcal{J}$ generated by $\{\Psi_1 = (V_1 \lrcorner \alpha), \Psi_2 = (V_2 \lrcorner \alpha)\}$ is nonsingular and admits a one-dimensional characteristic distribution $D(\mathcal{J})$; this is given by vector fields satisfying $X \lrcorner \alpha = 0$.*

Proof. As Ψ_1, Ψ_2 are both $N-1$ forms, $(X \lrcorner \Psi_j) \in \mathcal{J}$ is equivalent to $X \lrcorner \Psi_j = 0$. Thus the corollary is merely a restatement of lemma 4; notice this implies that the space of vector fields satisfying $X \lrcorner \alpha = 0$ has constant dimension, i.e. $\mathcal{J}$ is nonsingular. □

Theorem 4. *Let (M, g, ω_α) be a hyper-Kahler manifold of real dimension $4n$; let $\{\mathcal{H}^\alpha : M \to \boldsymbol{R}\}$ be three smooth functions. Let ϑ be the $(4n-1)$-form defined by (2), and let $\mathcal{J}$ be the nonsingular ideal associated with the variational principle defined by I. Then the characteristic distribution $D(\mathcal{J})$ for $\mathcal{J}$ is one dimensional and is generated by the hyper-Hamiltonian vector field Z defined by (3).*

Proof. Specialize lemma 4 and its corollary to the case $\alpha = \mathrm{d}\vartheta$, and use theorem 1. □

Remark 8. It follows from this that the vector field Z is everywhere tangent to integral manifolds of $\mathcal{J}$, i.e. to extremal sections for I. Moreover, by proposition A.1 (see the appendix), it also shows that the $(4n-1)$-dimensional extremal sections Φ for I can be described by assigning their value on a suitable $(4n-2)$-dimensional manifold and pulling them along integral curves of Z.

5. Integral invariants

The Poincaré invariants (Poincaré form and Poincaré–Cartan integral invariant) play a central role in the canonical structure of Hamiltonian mechanics. In hyper-Hamiltonian mechanics, we have objects enjoying the same properties; these turn out to be, respectively, the forms ϑ and φ introduced above (see (2)).

We consider as usual a differentiable manifold M of dimension $4n$ equipped with a hypersymplectic structure $\{\omega_\alpha\}$, and the extended phase space $M \times \boldsymbol{R}$.

Theorem 5. *Let γ_0 be a closed and oriented $(4n-1)$ submanifold of the extended phase space $M \times \boldsymbol{R}$; let γ_t be the manifold obtained by transporting γ along the flow of the hyper-Hamiltonian vector field Z defined in (3). Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \vartheta = 0. \tag{8}$$

Proof. Let Z_t denote the flow of Z. We have

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \vartheta = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_0} Z_t^* \vartheta = \int_{\gamma_0} \frac{\mathrm{d}}{\mathrm{d}t} (Z_t^* \vartheta) = \int_{\gamma_0} Z_t^* [\mathrm{d}(Z \,\lrcorner\, \vartheta) + Z \,\lrcorner\, \mathrm{d}\vartheta] = 0, \tag{9}$$

where the last integral vanishes because $Z \,\lrcorner\, \mathrm{d}\vartheta = 0$ and (by Stokes' theorem) because γ_0 is closed. □

We consider the special case of the construction considered above (leading to the Poincaré–Cartan invariant) in which the manifold γ_0 lies on a hyperplane at constant t.

This gives the Poincaré relative invariant, that should also be reinterpreted as the conservation of the volume form under the hyper-Hamiltonian flow.

Theorem 6. *Let γ_0 be a closed and oriented $(4n-1)$ submanifold of the extended phase space $M \times \boldsymbol{R}$ lying in the fibre over t_0 of the fibration $t : M \times \boldsymbol{R} \to \boldsymbol{R}$. Let γ_t be the manifold obtained by transporting γ along the flow of the hyper-Hamiltonian vector field Z defined in (3). Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\gamma_t} \varphi = 0. \tag{10}$$

Proof. In this case $\mathrm{d}t = 0$ on γ_t and the integration of ϑ on γ_t reduces to the integration of φ on the same manifold γ_t. Therefore (8) means that the integral of φ over γ_t is constant, i.e. (10). □

6. Hypersymplectic structures in $\boldsymbol{R}^4$

6.1. Standard structures

After developing the general theory in abstract terms, it will be useful to consider the simplest nontrivial example of hypersymplectic manifold. This is provided by $M = \boldsymbol{R}^4$ with standard Euclidean metric $g_{ij}(x) = \delta_{ij}$.

Despite the fact this is just a (simple) exercise of linear algebra, we shall explicitly write the hyper-Hamiltonian equations, and this for three reasons:

(a) this is the simplest case in which our construction applies, and having a fully explicit example can only help our understanding;
(b) the explicit formulation of the equation of motion in the standard case will be useful in section 7 when we discuss the quaternionic oscillator;

(c) discussion of this simple case will clarify some points which were not fully discussed above, referring instead to this explicit example.

The latter were first, the reason why we left the possibility that the orientation of the metric and the orientation of the symplectic structure disagree, and second, we use the standard structure to give an explicit example of a hyper-Hamiltonian vector field that is not Hamiltonian, whatever symplectic structure we define on M, showing that the dynamic that we propose is a real extension of Hamiltonian dynamics.

We use Cartesian coordinates x^i in $\boldsymbol{R}^4$, and the volume form will be $\Omega = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$. The space $\Lambda^2(\boldsymbol{R}^4)$ is six dimensional, and is spanned by

$$\begin{aligned}
\mu_1 &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4, & \eta_1 &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^2 \wedge \mathrm{d}x^4,\\
\mu_2 &= \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3, & \eta_2 &= \mathrm{d}x^4 \wedge \mathrm{d}x^1 + \mathrm{d}x^2 \wedge \mathrm{d}x^3,\\
\mu_3 &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^4 \wedge \mathrm{d}x^2, & \eta_3 &= \mathrm{d}x^2 \wedge \mathrm{d}x^1 + \mathrm{d}x^3 \wedge \mathrm{d}x^4.
\end{aligned}$$

Note that the μ span the space $\Lambda^2_+(M)$ of self-dual forms; the η span the space $\Lambda^2_-(M)$ of anti-self-dual forms.

Note also that the $\mu_\alpha \wedge \mu_\alpha = \Omega$, $\eta_\alpha \wedge \eta_a = -\Omega$. We shall refer to these as standard hypersymplectic structures of positive and negative type respectively. We also denote as $\mathcal{Q}_\pm$ the quaternionic structures spanned by these, and $\mathcal{S}_\pm$ their unit spheres. Obviously we have $\Lambda^2_\pm(M) = \mathcal{Q}_\pm$.

We write two-forms on $\boldsymbol{R}^4$ as $\omega = (1/2)(J)_{im}\, \mathrm{d}x^i \wedge \mathrm{d}x^m$, with J an antisymmetric tensor. We write the tensors corresponding to the μ_α as K_α, those corresponding to η_α as H_α (and their triples as $\boldsymbol{K}$ and $\boldsymbol{H}$).

Explicit expressions of these are as follows:

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\\ -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

$$H_1 = \begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 & 0 & -1\\ 0 & 0 & 1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad H_3 = \begin{pmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The complex structures are $Y_\alpha = g^{-1} J_\alpha$ and in the present case of a Euclidean metric we also write them as K_α and H_α (with a raised index).

Remark 9. The $\boldsymbol{K}$ and $\boldsymbol{H}$ span $su(2)$ algebras, which we denote as $su(2)_\pm$; they correspond to the left and right spinor algebras (note indeed that reversing the orientation of space exchanges the $\boldsymbol{K}$ and $\boldsymbol{H}$).

It is immediate to check that $[K_\alpha, H_\beta] = 0$ for all α, β; actually, if we look for the centralizer of $SU(2)_\pm$ in $GL(4, \boldsymbol{R})$, this is just $SU(2)_\mp$. This corresponds to the well known relation $so(4) \simeq su(2)_+ \oplus su(2)_-$, or equivalently to $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M) \equiv \mathcal{Q}_+ \oplus \mathcal{Q}_-$.

In the case of the standard positive-type hypersymplectic structure in $M = (\boldsymbol{R}^4, \delta)$, the equations of motion will be simply

$$\dot{x}^i = \sum_{\alpha=1}^{3} (K_\alpha)^{ij} \frac{\partial \mathcal{H}^\alpha}{\partial x^j}. \tag{11}$$

For the negative-type hypersymplectic structure the K_α are replaced by the H_α.

Lemma 5. *There are equations of the form (11) which are not Hamiltonian, whatever symplectic structure we define in M.*

Proof. To prove this, we use the following result from section 3 of [21]: *given a linear vector field $X = A^i{}_j x^j \partial_i$, if* $\mathrm{Tr}(A^{2k+1}) \neq 0$ *for some $k \in \boldsymbol{N}$, this is not Hamiltonian with respect to any symplectic structure.* Note that vanishing of $\mathrm{Tr}(A)$ corresponds to the condition of zero divergence, which is also satisfied by hyper-Hamiltonian flows.

Thus we only have to exhibit an example where $\mathcal{H}^\alpha = (1/2) D^\alpha_{ij} x^i x^j$ (with D^α symmetric matrices) and $A := \sum_\alpha K_\alpha D^\alpha$ satisfies $\mathrm{Tr}(A^3) \neq 0$. This is obtained for example if $\mathcal{H}^1 = (1/2)[(x^1)^2 - (x^2)^2 + (x^3)^2 - (x^4)^2 + 2(x^1x^4 - x^2x^3)]$, $\mathcal{H}^2 = (1/2)|x|^2$ and $\mathcal{H}^3 = 0$. □

We shall now consider the standard hypersymplectic structures defined above, and derive explicit expressions for the associated hyper-Hamiltonian dynamics. We consider the positive-type hypersymplectic structure.

It is immediate to check that, for any $\alpha = 1, 2, 3$,

$$\omega_\alpha \wedge \omega_\alpha \;=\; 2\,\Omega \;=\; 2\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4.$$

Obviously, with $X = f^i \partial_i$, we have

$$\begin{aligned} X \lrcorner\, \Omega = {} & f^1\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \;-\; f^2\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ & + f^3\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \;-\; f^4\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3. \end{aligned}$$

The computation of $\mathrm{d}\mathcal{H}^\alpha \wedge \omega_\alpha$ is also immediate, and we obtain

$$\begin{aligned} \sum_{\alpha=1}^{3} \mathrm{d}\mathcal{H}^\alpha \wedge \omega_\alpha = {} & (\partial_3\mathcal{H}^1 + \partial_1\mathcal{H}^2 - \partial_2\mathcal{H}^3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \\ & + (\partial_4\mathcal{H}^1 - \partial_2\mathcal{H}^2 - \partial_1\mathcal{H}^3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^4 \\ & + (\partial_1\mathcal{H}^1 - \partial_3\mathcal{H}^2 + \partial_4\mathcal{H}^3)\,\mathrm{d}x^1 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4 \\ & + (\partial_2\mathcal{H}^1 + \partial_4\mathcal{H}^2 + \partial_3\mathcal{H}^3)\,\mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4. \end{aligned}$$

The equations of motion are immediately obtained by comparing this and the previous expression; these are

$$\begin{aligned} \dot{x}^1 &= (\partial\mathcal{H}^1/\partial x^2) + (\partial\mathcal{H}^2/\partial x^4) + (\partial\mathcal{H}^3/\partial x^3) \\ \dot{x}^2 &= -(\partial\mathcal{H}^1/\partial x^1) + (\partial\mathcal{H}^2/\partial x^3) - (\partial\mathcal{H}^3/\partial x^4) \\ \dot{x}^3 &= (\partial\mathcal{H}^1/\partial x^4) - (\partial\mathcal{H}^2/\partial x^2) - (\partial\mathcal{H}^3/\partial x^1) \\ \dot{x}^4 &= -(\partial\mathcal{H}^1/\partial x^3) - (\partial\mathcal{H}^2/\partial x^1) + (\partial\mathcal{H}^3/\partial x^2). \end{aligned}$$

6.2. General structures

Let us now consider a general hypersymplectic structure $\boldsymbol{O}$; we denote a symplectic structure by ω and a point in $M = \boldsymbol{R}^4$ by x; let ω_x be the evaluation of ω in $x \in M$. The two lemmas below show that $\boldsymbol{O}$ is always equivalent to one of the two structures considered above.

Lemma 6. *If ω_x is unimodular, it belongs either to $\mathcal{S}_+$ or to $\mathcal{S}_-$.*

Proof. Let Y be the complex structure associated with ω. We work in coordinates, and write in full generality $Y(x) = (\boldsymbol{a} \cdot \boldsymbol{K}) + (\boldsymbol{b} \cdot \boldsymbol{H})$. Requiring ω to be unimodular, i.e. $Y^T Y = I$, we obtain two conditions: the vanishing of off-diagonal terms reads $a_\alpha b_\beta = 0\,\forall\,\alpha, \beta = 1, 2, 3$, so that $|\boldsymbol{a}| \cdot |\boldsymbol{b}| = 0$; setting diagonal terms equal to unity (together with the previous condition) yields moreover $|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 = 1$. □

Lemma 7. *A hypersymplectic structure in $M = \boldsymbol{R}^4$ is made either of positive-type symplectic structures, or of negative-type symplectic structures, but in no case by symplectic structures of the two kinds.*

Proof. A hypersymplectic structure corresponds to a $su(2)$ algebra in the way discussed above. As also mentioned above, the spans of the $\boldsymbol{K}$ and of the $\boldsymbol{H}$ correspond to $su(2)_\pm$ algebras, but no $su(2)$ algebra is generated by a mixture of matrices belonging to $su(2)_+$ and $su(2)_-$ algebras. □

6.3. Extension to 4n dimensions

We stress that the standard structures are immediately extended to structures in higher dimension. In the case $M = \boldsymbol{R}^{4n}$ (with Euclidean metric), take *block reducible* structures. By this we mean that $\omega_\alpha = (1/2)(J_\alpha)_{im}\, \mathrm{d}x^i \wedge \mathrm{d}x^m$, and the $Y_\alpha = g^{-1} J_\alpha$ generate a representation of the $su(2)$ algebra in $\boldsymbol{R}^{4n}$, which is the direct sum of irreducible representations on four-dimensional subspaces.

In this case the matrices acting on each four-dimensional block will be either $\boldsymbol{K}$ or $\boldsymbol{H}$. We thus have, in block notation, $J_\alpha = L_\alpha^{s_1} \oplus \cdots \oplus L_\alpha^{s_n}$, where $s_k = \pm$, and $L_\alpha^{(+)} = K_\alpha$, $L_\alpha^{(-)} = H_\alpha$ (notice that we could obtain equivalent hypersymplectic structures by orthogonal changes of variables on each $\boldsymbol{R}^4$ block). The analysis conducted in $\boldsymbol{R}^4$ does apply on each block.

7. Quaternionic oscillators

There is no need to stress the relevance and ubiquitous role of (harmonic and nonlinear) oscillators in standard Hamiltonian mechanics; we want to discuss here the hyper-Hamiltonian oscillators, with a view to the problem of integrable hyper-Hamiltonian systems (we assume the reader is familiar with Hamiltonian integrable systems).

Our intuitive understanding of hyper-Hamiltonian integrable systems will be that of systems which can be mapped to a system of hyper-Hamiltonian oscillators.

We shall consider systems in $M = \boldsymbol{R}^{4n}$ (with standard Euclidean metric), and standard (say positive-type) hypersymplectic structure; as the hyper-Kahler structure induces in this case a quaternionic structure, we shall speak of *quaternionic oscillators*.

We shall consider nontrivial systems with compact invariant manifolds, and start by discussing the case $n = 1$.

7.1. The standard four-dimensional case

The simplest nontrivial case of hyper-Hamiltonian dynamics is the one where we have quadratic Hamiltonians $\mathcal{H}^\alpha$, i.e. $\mathcal{H}^\alpha(x) = (1/2)c_\alpha |x|^2$, with c_α real constants; in this case we obtain $\dot{x}^i = c_\alpha K_{ij}^\alpha x^j$, which is easily integrated (see the more general discussion below).

Let us actually write $\rho \equiv (1/2)|x|^2$, and consider the class of nonlinear systems where $\mathcal{H}^\alpha(x) = \mathcal{H}^\alpha(\rho)$, i.e. assume the $\mathcal{H}^\alpha$ are arbitrary smooth functions of ρ. We call these *quaternionic oscillators*.

In this case, write $A^\alpha = \mathrm{d}\mathcal{H}^\alpha/\mathrm{d}\rho$; we have $\nabla\mathcal{H}^\alpha = A^\alpha(\rho)x$, and the equations of motion (1) read simply (see section 6)

$$\dot{x}^i = \sum_{\alpha=1}^{3} A_\alpha(\rho)\,(K_\alpha)^i{}_j\, x^j. \tag{12}$$

Notice that $\mathrm{d}\rho/\mathrm{d}t = \sum_{\alpha=1}^{3} A^\alpha(\rho)\,[x^i (K_\alpha)_{ij} x^j] = 0$; the last equality follows from $K_\alpha = -K_\alpha^T$. Therefore ρ (and hence $|x(t)|$) is a constant of motion under any hyper-Hamiltonian flow for Hamiltonians which are functions of ρ alone.

As $\rho(t) = \rho_0$, we can on any trajectory rewrite (9) as

$$\dot{x}^i = \sum_\alpha c_\alpha^0\,(K_\alpha)^i{}_j x^j \;=\; \nu_0 (K_\alpha)^i{}_j x^j, \tag{13}$$

where $c_\alpha^0 = A_\alpha(\rho_0)$, and

$$\nu_0 := \sqrt{(c_1^0)^2 + (c_2^0)^2 + (c_3^0)^2}, \qquad K^i{}_j \;=\; \frac{1}{\nu_0}\sum_{\alpha=1}^{3} c_\alpha^0\,(K_\alpha)^i{}_j.$$

The solution to (10) is obviously $x(t) = \exp[K\nu_0 t]x(0)$; expanding this in a power series in t and using $K^2 = -I$, we obtain immediately

$$x(t) = [\cos(\nu_0 t)I + \sin(\nu_0 t)K]x(0). \tag{14}$$

This represents a uniform motion on a great circle—identified by the vectors $\boldsymbol{x}_0 = x(0)$ and $\boldsymbol{x}_1 = Kx(0)$—of the sphere S^3 of radius $r_0 = |x(0)|$. The frequency ν_0 of such motions will be the same for motions on the same sphere: it depends only on the radius r_0 (note that the c_α^0 also depend on r_0).

Therefore any sphere S^3 of radius $r_0 \neq 0$ is covered by periodic circular motions, unless $\nu_0(r_0) = 0$, all of them with the same period $T_0 = 2\pi/\nu_0$; in this way the hyper-Hamiltonian flow (9) partitions S^3 into S^1 equivalence classes (the dynamical orbits) and thus realizes a Hopf fibration $S^3/S^1 = S^2$ of the three-sphere [11].

7.2. The (4n)-dimensional case

Let us move on to considering $M = \boldsymbol{R}^{4n}$, again with standard Euclidean metric; we shall use Cartesian coordinates $\{x^1, \ldots, x^{4n}\}$. We also define block variables $\{\xi_1, \ldots, \xi_n\}$ with $\xi_p \in \boldsymbol{R}^4$ corresponding to x coordinates in the pth block, $\xi_p^i = x^{4(p-1)+i}$ (where $i = 1, \ldots, 4$ and $p = 1, \ldots, n$); we also write $\rho_p = (1/2)|\xi_p|^2$.

We shall consider the case where $\boldsymbol{R}^{4n}$ is equipped with a standard block reducible hypersymplectic structure (see section 6), $J_\alpha = L_\alpha^{s_1} \oplus \cdots \oplus L_\alpha^{s_n}$. We have $L_\alpha^+ = K_\alpha$, $L_\alpha^- = H_\alpha$.

We shall now assume the Hamiltonians depend only on the ρ_p (we say we have a quaternionic n-oscillator):

$$\mathcal{H}^\alpha(x) = \mathcal{H}^\alpha(\rho_1, \ldots, \rho_n);$$

we write the Jacobian of the $\mathcal{H}$ with respect to the ρ variables as $A_p^\alpha := \partial\mathcal{H}^\alpha/\partial\rho_p$. In this case the equations of motion are (sum on repeated Greek and Latin indices will be implied, except for the block index p)

$$\dot{x}^i = (J_\alpha)^{ik}\partial_k \mathcal{H}^\alpha$$

and can be written as (no sum on p)

$$\dot{\xi}_p^i \;=\; A_p^\alpha(\rho_1, \ldots, \rho_n)(L_\alpha^{\sigma_p})^i{}_k \xi_p^k.$$

Again the ρ_p are constants of motion (no sum on p):

$$\frac{\mathrm{d}\rho_p}{\mathrm{d}t} = \frac{\mathrm{d}\xi_p^i}{\mathrm{d}t}\frac{\partial\rho_p}{\partial\xi_p^i} \;=\; A_p^\alpha(\rho_1, \ldots, \rho_n)[\xi_p^i (L_\alpha^{\sigma_p})^i{}_k \xi_p^k] = 0,$$

where the last equality follows from the antisymmetry of the $L_\alpha^{\sigma_p}$.

Hence the matrices A^α_p are constant under the flow. If we are given an initial datum $x(0)$, and thus the value of the constants of motion ($\rho_1 = b_1, \dots, \rho_n = b_n$), we can write

$$\dot{\xi}_{(p)} = \sum_{\alpha=1}^{3} c^\alpha_{(p)} (L^{\sigma_p}_\alpha) \xi_{(p)} = \nu_{(p)} L_{(p)} \xi_{(p)}, \tag{15}$$

where $c^\alpha_{(p)} = A^\alpha_p(b_1, \dots, b_n)$, and

$$\nu_p = \sqrt{(c^1_{(p)})^2 + (c^2_{(p)})^2 + (c^3_{(p)})^2}, \; L_{(p)} = (1/\nu_p) \sum_\alpha c^\alpha_{(p)} L^{\sigma_p}_\alpha .$$

That is, on each block we have the same situation as discussed in section 7.1; notice that the frequencies ν_p depend not only on the value b_p of ρ_p, but also on the values b_q of the other variables ρ_q ($p \neq q$).

8. Discussion: the relation between hyper-Hamiltonian and standard Hamiltonian integrability

We would like to discuss the relation between hyper-Hamiltonian integrability and standard Hamiltonian integrability for the class of systems considered here.

8.1. Dimension four

Let us first of all focus on the case given by $\mathcal{H}^1 = |x|^2/2$, $\mathcal{H}^2 = \mathcal{H}^3 = 0$; this corresponds to two uncoupled and identical harmonic oscillators with conserved energies $E_a = (1/2)[(x^1)^2 + (x^2)^2]$ and $E_b = (1/2)[(x^3)^2 + (x^4)^2]$.

The solutions of nonzero energy $E = E_a + E_b = r_0^2/2$ describe a circle S^1 lying on the sphere S^3 of radius r_0. When E_a and E_b are both nonzero (i.e. both oscillators are actually excited) these also lie on a torus $\boldsymbol{T}^2 \subset S^3$, and the circle S^1 corresponding to the solution is a combination of the two fundamental cycles of the torus.

The cases $E_a = 0$, $E_b \neq 0$ and $E_a \neq 0$, $E_b = 0$ correspond to degenerate situations in which the common level set of E_a and E_b is not a torus $\boldsymbol{T}^2$, but is reduced to a circle $\boldsymbol{T}^1 = S^1$, which is just the trajectory of the solution.

It should be recalled that the Hopf S^3 fibration can indeed be described as a singular fibration of S^3 in $\boldsymbol{T}^2$ tori, with two singular fibres, which correspond to the special cases in which all the energy is on one oscillator and the other is not excited; thus these two ways (hyper-Hamiltonian and standard Hamiltonian) of describing the situation are immediately related, as they should be.

Let us now return to the general (nonlinear) integrable case described by (9), whose solutions are given by (11); on each S^3 sphere of radius $r_0 \neq 0$, i.e. on each nonzero level manifold[5] for the energy $E = \rho$ we can indeed reduce to a two-oscillator description (see (9) and (10) above). Such a system is integrable in the Arnold–Liouville sense, since the set on which the fibration in tori is singular is of zero measure in the phase space.

However, it should be noticed that in considering this system as an integrable two-oscillator system we are completely overlooking the quaternionic structure of the system and of the whole class to which it belongs. Also, this system is strongly degenerate if seen in terms of two oscillators: indeed the two oscillators are in 1:1 resonance for all values of H, i.e. all values of the action variables $I_1 = E_a$ and $I_2 = E_b$. Such a degeneration is of course enforced by the quaternionic structure, and thus generic in the frame of ‘quaternionic oscillators’.

[5] In this case we can speak of energy level manifolds as the three Hamiltonians depend on a single scalar function ρ.

On the other hand, if we recognize the quaternionic structure and the fact that we need therefore only the global constant of motion ρ to guarantee integrability (see the above discussion, and the remarks below in this section), we have at once much stronger information on the structure of the system and also need an easier construction to guarantee integrability.

The situation is similar to that met when we represent a quaternion by a pair of complex numbers (or a complex number by a pair of real ones): this is possible and correct, but in this way we are overlooking an additional and relevant structure, which we must then introduce by suitable relations between complex (or real) quantities.

Thus, in order to guarantee integrability in the sense of standard Hamiltonian mechanics we need two constants of motion and we have to construct a system of two action and two angle coordinates; using the quaternionic structure we only need one constant of motion, i.e. ρ, and we have to construct a system of coordinates in which three 'angle-like' coordinates are associated with the 'action' coordinate ρ. By 'angle-like' we mean coordinates on the sphere S^3, which can be seen as a generalization from $S^1 \simeq \boldsymbol{C}^{\mathbf{1}}$ to S^3 of the angular coordinates of standard Hamiltonian mechanics; as $S^3 \simeq \boldsymbol{H}^{\mathbf{1}} \simeq SU(2)$ (here $\boldsymbol{H}$ is the quaternion field and $\boldsymbol{H}^{\mathbf{1}}$ the set of quaternions of unit norm), these are of quaternionic nature. We call them *spin coordinates*.

Notice that the evolutions along spin coordinates do not commute; thus the equivalent of the familiar integrable Hamiltonian evolution equations $\dot{I}_k = 0$, $\dot{\varphi}_k = \omega_k(I)$, related to the Abelian group $\boldsymbol{T}^2$, is now given by (9), (10) or, more intrinsically, by $\dot{I} = 0$ $(I \equiv \rho)$, $\dot{\psi} = \alpha(I)$, where ψ represents coordinates on the group $SU(2) \simeq S^3$, and $\alpha(I) \in su(2)$ is an element of the algebra $su(2)$, constant on each level set of $I \equiv \rho$. This more involved (and not separable) structure is unavoidable, due to the non-Abelian nature of $SU(2)$.

8.2. Higher dimension

In the standard Hamiltonian integrable case with m degrees of freedom, i.e. for m Hamiltonian oscillators (say all of them excited), we have invariant $\boldsymbol{T}^m$ tori, and the solutions will cover densely $\boldsymbol{T}^k \subset \boldsymbol{T}^m$ tori, with $k \leqslant m$ depending on the rational relations between the frequencies of different degrees of freedom on the given $\boldsymbol{T}^m$; in the hyper-Hamiltonian integrable case (for n quaternionic oscillators) we have a similar situation, as we now discuss.

First of all we remark that, since $\rho = (\rho_1, \ldots, \rho_n)$ are constants of motion, the common level sets of the ρ_p are invariant manifolds under the dynamics we are considering; these level sets $\rho^{-1}(b_1, \ldots, b_n)$ will be, when all the b_p are nonzero, manifolds

$$S^3 \times \cdots \times S^3 = \boldsymbol{V}^n;$$

notice that these $\boldsymbol{V}^n$ represent a generalization of tori: in the same way as $\boldsymbol{T}^n$ is the topological product of n (distinct) S^1 factors, $\boldsymbol{V}^n$ is the topological product of n (distinct) S^3 factors. If k out of the n numbers b_p are zero, the level set $\rho^{-1}(b_1, \ldots, b_n)$ will be a $\boldsymbol{V}^{n-k}$ manifold.

We shall denote the trajectory with initial datum $x(0)$ as $\gamma \subset \boldsymbol{R}^{4n}$. The previous discussion shows that the projection of γ to each $\boldsymbol{R}^4$ block, given by $\xi_{(p)}(t)$, will be periodic.

If all the frequencies $\{\nu_1, \ldots, \nu_n\}$ are rational with respect to each other, the full solution in $\boldsymbol{R}^{4n}$ will also be periodic, i.e. $\gamma \approx S^1$; if m degrees of freedom are excited (i.e. there are m nonzero b_p), this γ will also be a submanifold of the invariant manifold $\boldsymbol{V}^m = \rho^{-1}(b_1, \ldots, b_n)$.

If m degrees of freedom are excited and the frequencies corresponding to $b_p \neq 0$ split into $k \leqslant m$ sets, each ν_p being rational with respect to frequencies in the same set and irrational with respect to frequencies in different sets, the solutions γ will densely cover $\boldsymbol{T}^k$ tori.

These $\boldsymbol{T}^k$ will be submanifolds of $\boldsymbol{V}^m$, and we can always choose the generators S^1 of $\boldsymbol{T}^k$ so that each generator lies in a different generator S^3 of $\boldsymbol{V}^m$. Indeed each generator of $\boldsymbol{T}^k$ will be a linear combination of the projections of γ to the blocks corresponding to each rational subset of frequencies; we can choose it to be just a single projection and thus to be in a S^3 factor of $\boldsymbol{V}^m$.

9. Final remarks

In this final section we briefly present some additional remarks to put our work into perspective and mention directions of future development. We thank an unknown referee for raising the problem discussed in point 4 below.

(1) First of all, it should be stressed that here we were mostly interested in the local structure of this hyper-Hamiltonian dynamics, and we have not considered problems arising from the global structure of the hyper-Kahler manifold M on which it is defined. Locally, any such M is isomorphic to $\boldsymbol{R}^{4n}$, so we could have limited ourselves to considering these spaces (as in sections 6–8).

However, as in the standard Hamiltonian case, most of our construction will extend to more general hyper-Hamiltonian manifolds, so that in our general discussion (sections 1–5) we preferred to deal with a generic hyper-Kahler manifold, pointing out where our discussion requires to work chart by chart.

Focusing on local properties means, of course, that we are not concerned with the geometrically most interesting recent results on hyper-Kahler manifolds and their global structure (which is also relevant in connection with physics); see the references given in the introduction, and in particular [20], for an overview of these.

A fortiori we are not providing any new insight into hyper-Kahler geometry nor are we providing any new nontrivial hyper-Kahler manifold. We actually needed only the very basic definitions of hyper-Kahler geometry; we supposed that a hyper-Kahler manifold M is given, and we defined a dynamics on M related to the choice of three Hamiltonian functions.

(2) We should also note that no attempts to generalize Hamiltonian dynamics in the direction proposed here seem to be present in the rapidly growing literature on hyper-Kahler manifolds (mostly devoted to their geometry and construction of nontrivial examples). A somewhat orthogonal approach to a hyper-Kahler generalization of the structure of standard Hamiltonian mechanics, focusing on Poisson structures, was suggested by Xu [36].

(3) We also mention that in field theory considerations of multiple symplectic structures are suggested by covariance requirements and lie at the basis of the de Donder–Weyl formalism, as discussed in detail in [23], who call this 'multisymplectic field theory'; however, our approach (limited to mechanics) seems—at least at the present stage—not to be related to this theory.

(4) As mentioned in the introduction, a most important result in hyper-Kahler geometry concerns the construction of nontrivial hyper-Kahler manifolds via a moment map procedure starting from a (possibly trivial) hyper-Kahler manifold equipped with a Lie group action [10, 27]. It is natural to ask what happens when a (covariant) hyper-Hamiltonian dynamics is defined on the first manifold, i.e. how the dynamics descends to the quotient.

Let $(M, g; \omega_\alpha)$ be a hyper-Kahler manifold of dimension m. Assume that there is a compact Lie group G (we denote by $\mathcal{G}$ the Lie algebra of G and by $\mathcal{G}^*$ its dual) acting freely on M and preserving its metric g and the three-forms ω_α (thus acting triholomorphically); this defines three moment maps $\mu_\alpha : M \to \mathcal{G}^*$, or a map $\mu : M \to \mathcal{G}^* \otimes \boldsymbol{R}^3$. It is known [27] that the quotient metric on $N = \mu^{-1}(0)/G$ is hyper-Kahler. We denote by β_α the reduction of ω_α on N.

Let $\mathcal{H}^\alpha$ be three G-invariant smooth functions $\mathcal{H}^\alpha : M \to \boldsymbol{R}$, $\mathcal{H}^\alpha(x) = \mathcal{H}^\alpha(gx)$ for all $g \in G$ and $x \in M$, and X be the hyper-Hamiltonian vector field on M corresponding to these; we recall that X is given by $X = \sum_\alpha X_\alpha$ with X_α identified by $X_\alpha \lrcorner \omega_\alpha = \mathrm{d}\mathcal{H}^\alpha$ (no sum on α).

However, each X_α is a Hamiltonian vector field, generated by the Hamiltonian $\mathcal{H}^\alpha$, with respect to the symplectic structure ω_α. Thus, each X_α descends to a Hamiltonian vector field W_α on N, by standard symplectic reduction. In other words, for each α there is a smooth function $\mathcal{K}^\alpha : N \to \boldsymbol{R}$ such that $W_\alpha \lrcorner \beta_\alpha = \mathrm{d}\mathcal{K}^\alpha$ (no sum on α). This shows at once that X descends to a hyper-Hamiltonian vector field $W = \sum_\alpha W_\alpha$ on the hyper-Kahler quotient N.

(5) Physically, one should consider generalizations of the present approach in at least two directions: on the one hand, one would like to consider pseudo-Riemannian rather than Riemannian manifolds; on the other hand, one should consider the quantum version of the theory. It appears that both of these are feasible, and we shall report on these matters in a separate note.

(6) Finally, we would like to point out that the dynamics introduced here can be obtained in a completely different and quite interesting way. One can look at standard Hamilton equations in terms of complex analysis, and extend them from the complex to the quaternionic case; one obtains then exactly the equations introduced here, as discussed in [29].

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Appendix. Ideals of differential forms

In our discussion of the variational formulation of the hyper-Hamiltonian equations of motion, we used some concepts from the theory of ideals of differential forms (here called just ideals, for short). This is maybe less widely known than the other tools used in the paper, so we collect here some definitions for the convenience of the reader (see [15, 22] or [4] for further details).

Definition A.1. *Let M be a smooth N-dimensional manifold, and $\mathcal{J}_k \subset \Lambda^k(M)$, for $k = 0, \ldots, N$. The subset $\mathcal{J} = \bigcup_{k=0}^N \mathcal{J}_k \subset \Lambda(M)$ is said to be an* ideal of differential forms *iff (i) $\eta \in \mathcal{J}$, $\psi \in \Lambda(M)$, $\Rightarrow$ $\eta \wedge \psi \in \mathcal{J}$ and (ii) $\beta_1, \beta_2 \in \mathcal{J}_k$, $f^1, f^2 \in \Lambda^0(M)$, $\Rightarrow f^1\beta_1 + f^2\beta_2 \in \mathcal{J}_k$.*

Definition A.2. *Let $i : S \to M$ be a smooth submanifold of M; S is said to be an* integral manifold *of the ideal $\mathcal{J}$ iff $i^*(\eta) = 0$ for all $\eta \in \mathcal{J}$.*

The ideal $\mathcal{J}$ is said to be *generated* by the forms $\{\eta^{(\alpha)}, \alpha = 1, \ldots, r\}$ (with $\eta^{(\alpha)} \in \mathcal{J}$) if each $\varphi \in \mathcal{J}$ can be written as $\varphi = \sum_\alpha \rho_{(\alpha)} \wedge \eta^{(\alpha)}$ for a suitable choice of $\rho_{(\alpha)} \in \Lambda(M)$, $\alpha = 1, \ldots, r$. If $\mathcal{J}$ is generated by $\{\eta^{(\alpha)}, \alpha = 1, \ldots, r\}$, then $i : S \to M$ is an integral manifold for $\mathcal{J}$ iff $i^*(\eta^{(\alpha)}) = 0$ for all $\alpha = 1, \ldots, r$.

Given an ideal $\mathcal{J}$, we associate with any point $x \in M$ the subspace $D_x(\mathcal{J}) \subset \mathrm{T}_x M$ defined by $D_x(\mathcal{J}) := \{\xi \in \mathrm{T}_x M : \xi \lrcorner \mathcal{J}_x \subset \mathcal{J}_x\}$.

Definition A.3. *If $D_x(\mathcal{J})$ has constant dimension, the ideal $\mathcal{J}$ is said to be* non-singular, *and the distribution $D(\mathcal{J}) = \{D_x(\mathcal{J}), x \in M\}$ is its* characteristic distribution*; any vector field $X \in D(\mathcal{J})$ is said to be a* characteristic field *for $\mathcal{J}$.*

The following proposition is easy to prove (e.g. using the local coordinates introduced in section 44 of [15], or directly from definitions above) and is used in remark 8.

Proposition A.1. *Let $\mathcal{J}$ be generated by forms of degree k, and let $i : S \to M$ be an r-dimensional integral manifold of $\mathcal{J}$, with $r < k$. Let X be a characteristic vector field for $\mathcal{J}$, and let X be nowhere tangent to $i(S)$. Let φ_t be the local one-parameter group of diffeomorphisms generated by X. Then the $(r+1)$-dimensional manifold $(-\varepsilon, \varepsilon) \times S \ni (t, x) \mapsto \varphi_t(x) \in M$ is an integral manifold of $\mathcal{J}$.*

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